Visualizing Real Algebraic Curves and Surfaces with Singularities

Oliver Labs

Saarland University (Germany)
E-Mail: Labs@math.uni-sb.de, mail@OliverLabs.net.

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Introduction

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Another example:
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- Currently, a lot of work on visualization of curves/surfaces.
- E.g., attempts to do real-time visualization of surfaces.
- So: a need for good test examples.
- Current practice (often, not always): use random equations, curves/surfaces with (many) ordinary double points (the simplest singularities), . . .
- We want to be able to justify statements such as: we can visualize curves/surfaces up to degree 10.
- As an approach to this, we give a list of equations which correspond to curves/surfaces which are — in some natural sense — at least close to the most difficult ones.
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The Challenge

- Many of our examples may be visualized correctly using some existing software.
- However, in almost all cases, the computation is too slow for many applications.
- Even in the case of plane curves of moderate degree, to my knowledge, there is no software which produces a good and quick visualization in all cases without interaction (e.g. by changing some $\varepsilon$, sometimes even this does not help!).
- So, the challenge is often not the ability of visualizing an example correctly (at least not for plane curves), but to do it correctly in short time.
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Definition

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A real algebraic surface of degree \( d \) in \( \mathbb{R}^3 \) is (the zero-set of) a polynomial in three variables of degree \( d \):

\[
x^d + a_1 x^{d-1} y + a_2 x^{d-1} z + a_3 x^{d-1} + \ldots + a_k.
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Example

\begin{align*}
d &= 2 \\
d &= 3 \\
d &= 5
\end{align*}
Overview

Three parts of the talk:

- Exact input data (basically: rational coefficients).
- Data with noise / floating point coefficients.
- A new field of application: convex algebraic geometry.
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Why do we need good and quick visualizations?

Example (Visualization of the Swallowtail)

Consider the polynomial \( p \) in one variable \( x \) having 3 parameters: \( p = x^4 + ax^2 + bx + c \).

\( p \) has a double root iff \( p(x_0) = \frac{\partial p}{\partial x}(x_0) = 0 \) for some \( x_0 \).

\[ x_0^4 + ax_0^2 + bx_0 + c = 0, \quad 4x_0^3 + ax_0 + b = 0. \]

Eliminating \( x_0 \), we get a condition on \( a, b, c \) which holds iff \( p \) has a double root (the discriminant):

\[-4b^2c^3 - 16ac^4 + 27b^4 + 144ab^2c + 128a^2c^2 - 256a^3 = 0.\]

Moreover, the geometry of this surface tells us everything about the distribution of the roots of \( p \):

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surfex

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Some Surface Visualization Issues

Summary: Visualization Challenges

Some Uncertainty Issues

Visualizing Convex Algebraic Geometry

Conclusion
Some Visualization Issues

For the first part of the talk, we assume that the curve or surface we want to visualize is given by exact data.

We call a visualization correct if a pixel is colored iff the curve has a point inside the pixel:

(a) the curve
(b) wrong
(c) correct
Visualization Issue 1: Non-Reduced Varieties

\[(x^2 + y^2 - z^2 - 1)^2 = 0\]

A squarefree version of the polynomial can be computed to solve the problem.

Attention: With non-exact data this step is already non-trivial!
Visualization Issue 1: Non-Reduced Varieties

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Visualization Issue 2: Small connected components
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Using computer algebra, we can compute a point on each connected component, e.g. by computing resultants.
Visualization Issue 3: Real Lower-Dimensional Parts

Using computer algebra, one can compute good equations for the singular locus or project the surface down to a plane to simplify the problem.
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Real Lower-Dimensional Parts (II)

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Singularities

**Definition**

A singularity of a surface $S$ in $\mathbb{R}^3$ is a point $p \in S$, s.t. all the partial derivatives vanish:

$$\frac{\partial S}{\partial x}(p) = \frac{\partial S}{\partial y}(p) = \frac{\partial S}{\partial z}(p) = 0.$$
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(a): node \((A_1^-)\)

\[ x^2 - y^2 + z^2 = 0 \]

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Example

- nodes \((A_1)\)
- cusps \((A_2)\)
- non-isolated

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Singularities

(a): node \((A_1^-)\)
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(b): node (solitary, \(A_1^+)\)
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(d): cusp ($A_2^+$)

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(e): non-isolated
\[
(y - x^2)^2 - y^4 = 0
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- nodes \((A_1)\)
- cusps \((A_2)\)
- non-isolated
Some people simply measure the complexity of a singular point \( p \) by its multiplicity (i.e. the lowest degree monomial, if \( p = (0, 0) \), e.g.: \( x^2 + y^{32} \) (mult 2), \( x^3 - y^{32} \) (mult 3)).

However, this is a simplified point of view!

Three examples of plane curves with multiplicity two:

\[
x^2 - y^2 = 0
\]

\[
x^2 - y^8 = 0
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\begin{align*}
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Three examples of plane curves with multiplicity two:

- $x^2 - y^2 = 0$
- $x^2 - y^8 = 0$
- $x^2 - y^{32} = 0$
We have to look closer!

When should two singularities be considered \textit{equal}?

\exists \text{ coordinate change (local diffeomorphism)}?

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\[ x^3 - y^2 \quad \approx \quad x^5 - y^2 \]
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See: Arnold’s classification.
Invariants

We may distinguish a singularity at $(0, 0)$ of a plane curve $f$ by invariants (under local diffeomorphisms):

- **Multiplicity** (for the origin, this is the lowest order of the appearing terms), e.g.: $xy + y^3$ has mult. 2, $x^2 + z^{17}$ also,
- **Milnor number** (number of 'vanishing cycles'; computable as the multiplicity of $(0, 0)$ as a root of the system of polynomials $f_x, f_y$),
- **Tjurina number** (multiplicity of $(0, 0)$ as a root of the system of polynomials $f, f_x, f_y$),
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- **Dynkin Diagrams**,

- **Spectrum of a singularity** (eigenvalues of some operator).
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>>> These can be computed using SINGULAR (there is even a classification library)! Some also using other systems.
The Tangency

Definition

The **Tangency** $\text{tang}(c_1, c_2)$ between two halfbranches $c_1, c_2$ of a plane curve singularity $p$ is the order in which the distance between points on these halfbranches goes to 0:

$$\left| c_1(\varepsilon) - c_2(\varepsilon) \right| = O(\varepsilon^{\text{tang}(c_1, c_2)})$$

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Examples of Plane Curves with High Tangencies

- An ordinary double point has tangency 1.
- The tangencies of the two halfbranches $c_1, c_2$ of the normal forms $x^{k+1} + y^2 = 0$ of an $A_k^-$-singularity, have tangencies $\text{tang}(c_1, c_2) = \frac{k+1}{2} (= \frac{d}{2})$.
- The tangencies of the plane curves $f_{k,2,-}^2 (x, y) = (y - x^k)^2 - y^{2k}$ of degree $d = 2k$ are (as $A_{2k^2-1}$-singularities): $\text{tang}_{(0,0)}(f_{k,2,-}^2) = \frac{2k^2}{2} = \frac{d^2}{4}$.
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An Upper Bound

Fix the degree $d$. Then the singularities cannot get too bad:

- Each possible node (# = $\delta_s$) in a deformation of a singularity $s$ reduces the genus of a plane curve by 1.
- Formula: $\text{genus}(f_d) = \frac{(d-1)(d-2)}{2} - \sum_{s \in \text{Sings}} \delta_s$.
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Some Surface Visualization Issues

Summary: Visualization Challenges
    Plane Curves
    Surfaces

Some Uncertainty Issues

Visualizing Convex Algebraic Geometry

Conclusion

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Visualizing Real Curves and Surfaces
Visualization Challenges for Plane Curves

- many solitary points \((x^2 + y^2 = 0)\)
- higher solitary points \((x^{2k} + y^2 = 0, (y - x^k)^2 + y^{2k} = 0)\)
- smooth curves with many components
- smooth curves with nested ovals
- high tangencies at isolated singularities \(((y - x^k)^l - y^{kl} = 0, l = 2, A_{2k^2-1-\text{sing.}})\)
- high solitary points close to other components: \(((y - x^k)^2 + y^{2k}) \cdot (y^2 - x^2 + \frac{1}{10}) + y^{2k+2}, k = 3\)
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- high tangencies at isolated singularities (e.g., $A_k$-singularities)
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- surf (misses solitary points)
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Summary: Visualization Challenges

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Up to now: exact input data assumed.

But:
- In many applications: no exact input data!
- Maybe bounds on the coefficients!

We thus need algorithms for computing...
- Good estimates for the type of singularity of which the data might be a small deformation.
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Oliver Labs, Saarland University (Germany)
Introduction

with Ph. Rostalski.

A **spectrahedron** in $\mathbb{R}^n$ is the set of all $x_1, \ldots, x_n$, such that

\[(*) \quad A_0 + x_1 A_1 + \cdots + x_n A_n \geq 0 \quad (\text{i.e. positive semi-definite})\]

for given symmetric $d \times d$ matrices $A_i$.

**Semidefinite programming (SDP)** is the computational problem of maximizing a linear function over a spectrahedron:

$$c_1 x_1 + \cdots + c_n x_n \quad \text{subject to} \quad (*) .$$

There are good algorithms for solving such problems.
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is:

- a convex set,
- bounded by parts of an algebraic surface of degree $d$,
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  \[ A(x) \geq 0 \iff \text{all principal minors } \geq 0, \]
- fewer inequalities:
  take all $1$, $2$, $\ldots$, $d$ polars w.r.t. an interior point.
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Examples

Cutting out the convex region by the polars:

- Polar illustration
- Pillow
- Plane curves
- Degree 3
- Eight singularities
- Degree 4

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Examples

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Cutting out the convex region by the polars:

\[
\begin{pmatrix}
1 & x & y \\
x & 1 & z \\
y & z & 1
\end{pmatrix} \geq 0
\]

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  - optimize in enough directions
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Conclusion
Conclusion

It is not easy to visualize plane curves correctly!
Conclusion

There is a list of explicit equations covering these cases (IMA publ., 2009)!
Conclusion

Singular points of curves are especially difficult!
Conclusion

Also for these cases, there is a list of explicit examples!
Conclusion

For surfaces, everything is even harder!

Also for surfaces, there have a list of explicit equations (pre-preprint available).
Conclusion

In particular, if lower-dimensional components exist!
Conclusion

If uncertainty about the input coefficients occurs, all questions become even more difficult!

See Joab Winkler’s talk for finding a reasonable solution nearby in the one-variable case.

An analogue in more variables which is better and faster than Z. Zeng’s approach would be very nice.
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Good news: new problems are coming up!

E.g.: Visualization of convex algebraic geometry brings new methods into the game!
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To finish: It is almost impossible to separate the topics of this workshop — they all add a piece to the big picture:

- Geometric modelling,
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- Approximate polynomial algebra.

For solving a problem in geometric modelling, we need polynomial algebra, and vice versa. And if we want real-time visualizations, we often have to work with approximate algebra, even if the initial data was exact. And even if we have approximate data, it might help to study these problems exactly (e.g., using interval arithmetic).
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Thank You

Thank you for your attention.

Oliver Labs

for related publications, see:
www.OliverLabs.net

for software and visualizations, see:
www.surfex.AlgebraicSurface.net
www.Calendar.AlgebraicSurface.net
www.imaginary-exhibition.com